Journal of Algebra 323 (2010) 1758–1764

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the B-canonical splittings of flag varieties

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ARTICLE INFO

Article history:
Received 4 September 2009
Available online 16 December 2009
Communicated by Peter Littelmann

Keywords:
Frobenius splitting
Flag varieties
Schubert varieties

ABSTRACT

Let G be a semisimple algebraic group over an algebraically closed field of positive characteristic. In this note, we show that an irreducible closed subvariety of the flag variety of G is compatibly split by the unique canonical Frobenius splitting if and only if it is a Richardson variety, i.e. an intersection of a Schubert and an opposite Schubert variety.

1. Introduction

Let G be a semisimple simply-connected algebraic group over an algebraically closed field of positive characteristic and let $B \subseteq G$ be a Borel subgroup. By work of Mathieu [6] and Ramanathan [8] (cf. also [3], §2.3, §§4.1–4.2 and [10]), the flag variety G/B admits a unique B-canonical splitting that compatibly splits all intersections of Schubert and opposite Schubert varieties (the so-called Richardson varieties). In this note, we show that the converse holds: Any irreducible closed subvariety of G/B that is compatibly split by ψ is a Richardson variety.

Here is an outline of the proof. We first show in Theorem 2.11 that if φ is a splitting of a normal variety X such that φ is a $(p-1)$st power and such that the divisor of zeroes $Z(\varphi)$ generates a normal intersection compatible system $\mathfrak{X}$ (cf. Definitions 2.2 and 2.8 below), then a closed irreducible subvariety $Z \subseteq X$ is compatibly split by φ if and only if $Z \in \mathfrak{X}$. Furthermore, let ψ denote the unique B-canonical splitting of G/B; then ψ is a $(p-1)$st power and $Z(\psi)$ generates an intersection compatible system whose elements are precisely the Richardson varieties. It is straightforward to check (cf. Theorem 3.3) that the set of Richardson varieties is a normal system. The result then follows immediately.

This note is based on a question asked by Allen Knutson. After I obtained this result, he informed me that he has independently obtained the same result. I would also like to thank Michel Brion, Shrawan Kumar, and George McNinch for helpful comments and discussions.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.12.004

2. Splitting facts

2.1. Review of Frobenius splitting methods

In this section we review the theory of Frobenius splitting. The main references are [3] and the seminal paper [7]. In the sequel we assume that all varieties are over an algebraically closed field k of positive characteristic p.

Let X be a scheme over k. We define a morphism F_X of schemes over $\mathbb{F}_p$ as follows. Set $F_X(x) = x$ for all $x \in X$ and define $F_X^{\#} : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ to be the pth power map $f \mapsto f^p$; this is clearly an $\mathbb{F}_p$-linear map. Note that F_X is not a morphism of schemes over $\overline{\mathbb{F}}_p$. This morphism is called the *absolute Frobenius morphism*. When the context is clear we'll drop the subscript and just write F.

Definition 2.1. We say that X is *Frobenius split* if there is an $\mathcal{O}_X$-linear map $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^{\#}$ is the identity map on $\mathcal{O}_X$.

Let $\mathcal{E}nd_F(\mathcal{O}_X)$ denote the sheaf of $\mathcal{O}_X$-linear maps $F_*\mathcal{O}_X \to \mathcal{O}_X$. Then $\mathcal{E}nd_F(\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules, where the $\mathcal{O}_X$-action is given by $(f.a)(h) = f \cdot (a(h))$ for $f \in \mathcal{O}_X$, $h \in F_*\mathcal{O}_X$, and $a \in \mathcal{E}nd_F(\mathcal{O}_X)$. Set $\mathrm{End}_F(X) := H^0(X, \mathcal{E}nd_F(\mathcal{O}_X))$.

Let $\varphi \in \mathrm{End}_F(X)$ be a splitting of X. We say that a closed subvariety $Y \subseteq X$ is *compatibly split* by φ if $\varphi(F_*\mathcal{I}) \subseteq \mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf of Y. We have the following useful facts:

(i) $\varphi \in \mathrm{End}_F(X)$ is a splitting of X if and only if $\varphi|_U$ it is a splitting of U, for any dense open subvariety $U \subseteq X$.
(ii) If $Y \subseteq X$ is a closed subvariety that is compatibly split by a splitting φ of X, then each irreducible component of Y is also compatibly split.
(iii) If Y and Z are closed subvarieties of X that are compatibly split by a splitting φ of X, then the set-theoretic intersection $Y \cap Z$ and union $Y \cup Z$ are also compatibly split.

Definition 2.2. Let X be a normal variety. Let X^{sing} denote the singular locus of X and let X^{reg} denote the regular locus. Recall that for all $n \in \mathbb{Z}$ we have $\omega_X^n = i_*\omega_{X^{\mathrm{reg}}}^n$, where $i : X^{\mathrm{reg}} \hookrightarrow X$ is the inclusion. Let $s \in H^0(X, \omega_X^{1-p})$; then we set $Z(s) := \overline{Z(s|_{X^{\mathrm{reg}}})}$, where by $Z(s|_{X^{\mathrm{reg}}})$ we mean the (set-theoretic) zero set of $s|_{X^{\mathrm{reg}}}$. Also, we say that s is a $(p-1)$st power if the restriction of s to X^{reg} is a $(p-1)$st power.

If X is an H-scheme for an algebraic group H, then there is a natural action of H on $\mathrm{End}_F(X)$ given by

$$(h.\varphi)(f) = h\big(\varphi\big(h^{-1}f\big)\big),$$

for all $h \in H$, $\varphi \in \mathrm{End}_F(X)$, and $f \in F_*\mathcal{O}_X$.

Theorem 2.3. *Let X be a normal variety. Then there is an $\mathcal{O}_X$-module isomorphism $\mathcal{E}nd_F(\mathcal{O}_X) \xrightarrow{\sim} \omega_X^{1-p}$. Furthermore, if X is an H-variety for an algebraic group H, then the induced isomorphism $\mathrm{End}_F(X) \xrightarrow{\sim} H^0(X, \omega_X^{1-p})$ is compatible with the H-structure on $\mathrm{End}_F(X)$ defined above and the natural H-structure on $H^0(X, \omega_X^{1-p})$.*

Using this theorem, we will from now on consider Frobenius splittings to be elements of $H^0(X, \omega_X^{1-p})$.

We also recall the following facts (cf. [3], Exercises 1.3.E (3) and (4)): Assume that X is normal and let D be a closed subvariety of pure codimension 1. Similarly to above, we set

$$\omega_X^{1-p}\big((1-p)D\big) := i_*\omega_{X^{\mathrm{reg}}}^{1-p}\big((1-p)\big(D \cap X^{\mathrm{reg}}\big)\big),$$

where $i : X^{\mathrm{reg}} \hookrightarrow X$ is the inclusion. Let $\varphi \in H^0(X, \omega_X^{1-p})$ be a splitting of X. Then we have:

(i) D is compatibly split by φ if and only if φ is in the image of the natural morphism

$$H^0\big(X, \omega_X^{1-p}\big((1-p)D\big)\big) \hookrightarrow H^0\big(X, \omega_X^{1-p}\big).$$

Further, if φ compatibly splits D then the degree of vanishing of φ on D is exactly $p-1$.

(ii) If φ is a $(p-1)$st power then D is compatibly split if and only if $D \subseteq Z(\varphi)$.

(Although these results are stated in [3] only for the smooth case, it is easy to check that the results extend to the normal case.)

We will need the following result from [5].

Proposition 2.4. *(See [5], Proposition 2.1.) Let X be a smooth variety and let $\varphi \in H^0(X, \omega_X^{1-p})$ be a splitting of X. If $Y \subsetneq X$ is compatibly split by φ, then $Y \subseteq Z(\varphi)$.*

2.2. Three lemmas on splittings of normal varieties

We will need the three following technical results.

Lemma 2.5. *Let X be a normal irreducible variety and let $\mathcal{L}$ be a line bundle on X. Let $n \in \mathbb{Z}^+$ and let $s \in H^0(X, \mathcal{L}^n)$ be such that $s|_U$ is an nth power on some nonempty open set $U \subseteq X$. Then s is an nth power.*

Proof. Let $t \in H^0(U, \mathcal{L}|_U)$ be such that $t^n = s|_U$. Since s has no poles on $X \setminus U$, neither does t. Hence, as X is normal, t extends to a global section $a \in H^0(X, \mathcal{L})$. Since $a^n|_U = s|_U$, we must have $a^n = s$ and hence s is an nth power. $\square$

Lemma 2.6. *Let X be a normal variety. Let $\psi \in H^0(X, \omega_X^{1-p})$ be a splitting and let $D \subseteq X$ be a compatibly split irreducible normal prime divisor. If ψ is a $(p-1)$st power, then so is the induced splitting of D.*

Proof. Since X is normal, $D^{\mathrm{reg}} \cap X^{\mathrm{reg}} \neq \emptyset$. By Lemma 2.5, it suffices to check that the restriction of ψ from X^{reg} to $D^{\mathrm{reg}} \cap X^{\mathrm{reg}}$ is a $(p-1)$st power, so we may assume that X and D are smooth. The result now follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
H^0(X, \omega_X^{-1}(-D)) & \xrightarrow{\otimes(p-1)} & H^0(X, \omega_X^{1-p}((1-p)D)) \\
\downarrow & & \downarrow \\
H^0(D, \omega_D^{-1}) & \xrightarrow{\otimes(p-1)} & H^0(D, \omega_D^{1-p})
\end{array}$$

where the horizontal arrows are the natural maps induced by the isomorphism $\omega_X^{-1}(-D)|_D \cong \omega_D^{-1}$. $\square$

For any $s \in H^0(X, \omega_X^{1-p}((1-p)D))$, we let (as above) $Z(s)$ denote the closure of $Z(s|_{X^{\mathrm{reg}}})$; note that if $\omega_X^{1-p}((1-p)D)$ is a line bundle this agrees with the usual definition of the zero set of s. Recall the following result ([1], Proposition 16.4): If X is a normal variety and D is a normal subvariety of X of pure codimension 1 such that $\omega_X^{1-p}((1-p)D)$ is a line bundle on X, then $\omega_X^{1-p}((1-p)D)|_D \cong \omega_D^{1-p}$.

Lemma 2.7. *Let X be a normal, irreducible variety and let $D \subseteq X$ be a normal prime divisor such that $\omega_X^{1-p}((1-p)D)$ is a line bundle on X. Let*

$$p : H^0\big(X, \omega_X^{1-p}\big((1-p)D\big)\big) \to H^0\big(D, \omega_D^{1-p}\big)$$

be the map induced by the isomorphism $\omega_X^{1-p}((1-p)D)|_D \cong \omega_D^{1-p}$ and choose $s \in H^0(X, \omega_X^{1-p}((1-p)D))$. Then $Z(p(s)) = Z(s) \cap D$.

Proof. By passing to a small open set, we may assume that X is affine and that $\omega_X^{1-p}((1-p)D)$ trivializes on X. Let $\theta \in H^0(X, \omega_X^{1-p}((1-p)D))$ be a trivialization; then $\theta|_D$ is a trivialization of $H^0(D, \omega_D^{1-p})$. Thus, using θ, the map p identifies with a surjection $p' : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_D$. Since p is just section restriction, p' is the natural restriction of functions; but then the result follows from the corresponding result for functions. $\square$

2.3. Systems of subvarieties

Definition 2.8. Let X be a variety. We say that a set $\mathfrak{X}$ of closed irreducible subvarieties of X is *intersection compatible* if the set of finite unions of elements of $\mathfrak{X}$ is closed under set-theoretic intersection. If A is a set of closed subvarieties of X, let $\mathfrak{S}(A)$ denote the minimal intersection-compatible system containing A. That is, $\mathfrak{S}(A)$ is the system obtained by iteratively taking set-theoretic intersections and irreducible components.

For any intersection compatible system $\mathfrak{X}$ and any $Y \in \mathfrak{X}$, set

$$\mathfrak{X}^Y := \{Y\} \cup \mathfrak{S}\big(\{E \in \mathfrak{X}\colon\ E \text{ is a divisor of } Y\}\big),$$

an intersection compatible subsystem of $\mathfrak{X}$. Also set $\overline{\mathfrak{X}^Y} := \{Z \in \mathfrak{X}\colon\ Z \subseteq Y\}$, another intersection compatible subsystem of $\mathfrak{X}$. Clearly $\mathfrak{X}^Y \subseteq \overline{\mathfrak{X}^Y}$.

We say that an intersection compatible system $\mathfrak{X}$ is *normal* if: (1) All elements of $\mathfrak{X}$ are normal; (2) for all $Y \in \mathfrak{X}$ we have $\mathfrak{X}^Y = \overline{\mathfrak{X}^Y}$; and (3) for all $Y \in \mathfrak{X}$ such that $Y^{\text{sing}} \neq \emptyset$ and for every irreducible component Z of Y^{sing}, there is a prime divisor D of Y such that $Z \subseteq D$ and $D \in \mathfrak{X}$. Note that (2) implies that for every $Y \in \mathfrak{X}$, $\mathfrak{X}^Y$ is a normal intersection compatible subsystem of $\mathfrak{X}$.

Given a Frobenius splitting ψ of a variety X, let D be the union of all prime divisors compatibly split by ψ and set $\mathfrak{X}_\psi := \{X\} \cup \mathfrak{S}(D)$. In this case, the set of finite unions of elements of $\mathfrak{X}_\psi$ is closed even under scheme-theoretic intersection, not just set-theoretic intersection.

Remark 2.9. Although it is clear that every element of $\mathfrak{X}_\psi$ is a compatibly split subvariety, I don't know if the converse holds in general; although see Theorem 2.11 below for a partial converse.

Lemma 2.10. *Let X be a normal variety and let $\psi \in H^0(X, \omega_X^{1-p})$ be a splitting of X such that ψ is a $(p-1)$st power and $\mathfrak{X}_\psi$ is a normal intersection compatible system. Let $D \subseteq X$ be a compatibly split prime divisor and let φ be the induced splitting of D. Then we have $\mathfrak{X}_\varphi = \mathfrak{X}_\psi^D$. In particular, $\mathfrak{X}_\varphi$ is also a normal system.*

Proof. Set $\mathfrak{Y} := \{Z \in \mathfrak{X}_\psi\colon\ Z \text{ is a divisor in } D\}$. Since φ is a $(p-1)$st power, it suffices to check that $\mathfrak{Y}$ is precisely the set of irreducible components of $Z(\varphi)$. Since ψ compatibly splits D, it is in the image of the natural map $\omega_X^{1-p}((1-p)D) \hookrightarrow \omega_X^{1-p}$, and we will think of ψ as a section of $\omega_X^{1-p}((1-p)D)$. Let $Z'(\psi)$ denote the zero set of ψ considered as a section of $\omega_X^{1-p}((1-p)D)$; then $Z'(\psi)$ is the union of all irreducible components of $Z(\psi)$ other than D.

We first check that each irreducible component of $Z(\varphi)$ is in $\mathfrak{Y}$. If $Z(\varphi) = \emptyset$ the result is trivial, so assume that $Z(\varphi) \neq \emptyset$. Let E be an irreducible component of $Z(\varphi)$. Assume, to the contrary, that for every compatibly split prime divisor $D' \neq D$ of X we have that E is not an irreducible component of $D' \cap D$. Since ψ is a $(p-1)$st power, this implies that E is not an irreducible component of $D' \cap D$ for any irreducible component D' of $Z(\psi)$. In particular, $E \not\subseteq Z'(\psi)$. Replacing X with $X \setminus Z'(\psi)$, we restrict to the case where $Z'(\psi) = \emptyset$. Hence ψ provides a trivialization $\omega_X^{1-p}((1-p)D) \xrightarrow{\sim} \mathcal{O}_X$, so by Lemma 2.7, $Z(\varphi) = Z'(\psi) \cap D = \emptyset$, which is false (note that D is normal since $\mathfrak{X}_\psi$ is a normal system). Hence $E \in \mathfrak{Y}$.

Conversely, to see that $\mathfrak{Y} \subseteq Z(\varphi)$, let $E \in \mathfrak{Y}$; then, as E is compatibly split by ψ, it is also compatibly split by φ, and the result obtains since $E \cap D^{\mathrm{reg}} \neq \emptyset$. $\square$

Theorem 2.11. *Let X be a normal variety and let ψ be a splitting of X such that ψ is a $(p-1)$st power and $\mathfrak{X}_\psi$ is a normal intersection compatible system. Then a closed irreducible subvariety Z of X is compatibly split by ψ iff $Z \in \mathfrak{X}_\psi$.*

Proof. The "if" part is clear. We prove the converse by induction on $\dim X$. If $\dim X = 0$ then the result is trivial; now assume that the result holds for all normal varieties of dimension $< n$ that satisfy the conditions of the theorem. Let X be a normal variety of dimension n and let ψ be a splitting of X such that ψ is a $(p-1)$st power and $\mathfrak{X}_\psi$ is a normal system. Let $Z \subseteq X$ be compatibly split by ψ.

I first claim that there is $D \in \mathfrak{X}_\psi$ such that D is a divisor of X and $Z \subseteq D$. If $Z \cap X^{\mathrm{reg}} \neq \emptyset$ then $Z \cap X^{\mathrm{reg}} \subseteq Z(\psi)$ by Proposition 2.4, and hence there is an irreducible component D of $Z(\psi)$ such that $Z \subseteq D$. Since ψ is a $(p-1)$st power, D is compatibly split by ψ, so $D \in \mathfrak{X}_\psi$, as desired. On the other hand, if $Z \subseteq X^{\mathrm{sing}}$, let E be an irreducible component of X^{sing} containing Z. Since $\mathfrak{X}_\psi$ is a normal system there is a prime divisor D of X such that $Z \subseteq E \subseteq D$ and $D \in \mathfrak{X}_\psi$. Hence the claim holds.

Let φ denote the induced splitting of D; we now obtain an intersection compatible system $\mathfrak{X}_\varphi$ on D. By Lemma 2.6, φ is a $(p-1)$st power, and by Lemma 2.10, $\mathfrak{X}_\varphi = \mathfrak{X}_\psi^D$ is a normal intersection compatible system containing Z. The result now follows by the induction hypothesis. $\square$

3. Compatibly split subvarieties of flag varieties

Let G be a semisimple simply-connected algebraic group over k. Fix a maximal torus $T \subseteq G$ and a Borel subgroup B of G containing T. Recall that the *weights* of G are the algebraic group homomorphisms $T \to k^*$. For any weight λ of G let $\mathcal{L}(\lambda)$ denote the equivariant bundle $G \times^B k_{-\lambda}$ on G/B with fiber $k_{-\lambda}$, and set

$$H^0(\lambda) := H^0\big(G/B, \mathcal{L}(\lambda)\big).$$

Let ρ denote the half-sum of the positive roots of G and set $\mathrm{St} := H^0((p-1)\rho)$, the *Steinberg module*.

Definition 3.1. A Frobenius splitting ψ of a B-variety X is called *B-canonical* if there is a B-equivariant morphism $\theta : \mathrm{St} \otimes k_{(p-1)\rho} \to H^0(X, \omega_X^{1-p})$ such that $\psi = \theta(f_- \otimes f_+)$, where $f_- \in \mathrm{St}$ is any nonzero lowest weight vector and $f_+ \in k_{(p-1)\rho}$ is any nonzero vector.

For any w in the Weyl group W of G set $C_w := BwB/B$, the Schubert cell in G/B corresponding to w and let $X_w := \overline{C_w}$ denote the Schubert variety corresponding to w. We similarly have the opposite Schubert cell $C^w := B^- wB/B$ and opposite Schubert variety $X^w := \overline{C^w}$. For any pair $v, w \in W$ set $C_w^v := C_w \cap C^v$ and $X_w^v := X_w \cap X^v$; these varieties are called *Richardson varieties*.

The following is essentially due to Mathieu [6] (see the comments following Lemma 2.3); see also §2.3 and §4.1 in [3] and §4.3 in [10].

Theorem 3.2. *There is a unique B-canonical ψ splitting of G/B. Moreover, ψ is a $(p-1)$st power, and $Z(\psi)$ is the union of the codimension-1 Schubert and opposite Schubert varieties. In particular, $\mathfrak{X}_\psi$ is precisely the set of all Richardson varieties.*

Proof. The existence and uniqueness of ψ is known (see the references above). We now check that ψ is a $(p-1)$st power. Indeed, let $m : H^0(\rho) \otimes H^0(\rho) \to H^0(2\rho)$ be the multiplication map; then $\psi = m(v_+ \otimes v_-)^{\otimes(p-1)}$, where v_+ (resp. v_-) is a nonzero highest (resp. lowest) weight vector in $H^0(\rho)$. Finally, the facts about $Z(\psi)$ and $\mathfrak{X}_\psi$ follow from the proof of Theorem 2.3.1 in [3]. □

Parts (i), (ii), and (iii) of the following theorem are just a restatement of Theorem 3.2 in [9] and Lemma 1 of [4].

Theorem 3.3. *Let $v, w \in W$.*

(i) *X_w^v is nonempty if and only if $v \leqslant w$, in which case X_w^v is a normal irreducible variety of dimension $l(w) - l(v)$. Furthermore, $X_{w'}^{v'} \subseteq X_w^v$ if and only if $v \leqslant v' \leqslant w' \leqslant w$.*
(ii) *C_w^v is an open nonsingular subvariety of X_w^v.*
(iii) *The boundary $\partial X_w^v := X_w^v \setminus C_w^v$ of X_w^v is a union of Richardson varieties.*

In particular, the set of Richardson varieties forms a normal intersection compatible system.

Proof. We need to verify the "in particular" part. By (i) each Richardson variety is normal, so part (1) of the definition of a normal system is satisfied. By (ii) and (iii), the singular locus of a Richardson variety is contained in a union of Richardson varieties, and by (i) every Richardson subvariety of a Richardson variety X_w^v is contained in a Richardson divisor of X_w^v. Thus part (3) of the definition of a normal system follows. We now need to check part (2) of the definition of a normal system.

We show, by induction on dimension, the following: Let Y be a Richardson variety and let Z be the union of all Richardson varieties that are divisors in Y. Then $\mathfrak{S}(Z)$ is the set of all Richardson varieties contained in Z. This is trivial for 0-dimensional Richardson varieties, so now choose $v, w \in W$ with $v < w$ and assume that the induction hypothesis holds for all Richardson varieties of dimension $< l(w) - l(v)$. We need to check the following:

(∗) For all Richardson divisors D in X_w^v, and for any Richardson divisor E of D, there is a Richardson divisor D' of X_w^v such that E is an irreducible component of $D' \cap D$.

Now, the Richardson divisors of X_w^v are the X_w^a for all $a > v$ with $l(a) = l(v) + 1$ and the X_b^v for all $b < w$ with $l(b) = l(w) - 1$. We'll verify that (∗) holds in the case $D = X_b^v$; a similar argument (or translation by w_0) will then give (∗) for the case $D = X_w^a$.

Fix $b < w$ with $l(b) = l(w) - 1$. We first consider the divisor $X_{b'}^v$ of X_b^v for $b' < b$, $l(b') = l(b) - 1 = l(w) - 2$. By Lemma 10.3 in [2], there exists $x \in W$ with $x \neq b$ and $b' < x < w$. Hence $X_{b'}$ is an irreducible component of $X_b \cap X_x$, so that $X_{b'}^v$ is an irreducible component of $X_x^v \cap X_b^v$. Since X_x^v is a divisor in X_w^v, the result now follows for $X_{b'}^v$.

Now consider the divisor $X_b^{v'}$ of X_b^v for $v' > v$, $l(v') = l(v) + 1$. Then

$$X_w^{v'} \cap X_b^v = X_w \cap X_b \cap X^{v'} \cap X^v = X_b^{v'}.$$

Since $X_w^{v'}$ is a divisor in X_w^v, the result now follows, and the proof is complete. □

We now have the main result.

Theorem 3.4. *Let $\psi \in H^0(G/B, \omega_{G/B}^{1-p})$ be the unique B-canonical splitting of G/B. Then a closed subvariety of G/B is compatibly split by ψ if and only if it is a union of Richardson varieties.*

Proof. This is immediate from Theorems 2.11, 3.2, and 3.3. □

References

[1] Flips and abundance for algebraic threefolds, in: Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque 211 (1992), Société Mathématique de France, Paris, 1992.
[2] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, in: Lie Groups and Their Representations, Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971, Halsted, New York, 1975, pp. 21–64.
[3] M. Brion, S. Kumar, Frobenius Splitting Methods in Geometry and Representation Theory, Progr. Math., vol. 231, Birkhäuser, Boston, 2005.
[4] M. Brion, V. Lakshmibai, A geometric approach to standard monomial theory, Represent. Theory 7 (2003) 651–680 (electronic).
[5] S. Kumar, V.B. Mehta, Finiteness of the number of compatibly split subvarieties, arXiv:0901.2098.
[6] O. Mathieu, Filtrations of G-modules, Ann. Sci. Ecole Norm. Sup. (4) 23 (4) (1990) 625–644.
[7] V.B. Mehta, A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, Ann. of Math. (2) 122 (1) (1985) 27–40.
[8] A. Ramanathan, Equations defining Schubert varieties and Frobenius splitting of diagonals, Publ. Math. Inst. Hautes Etudes Sci. 65 (1987) 61–90.
[9] R.W. Richardson, Intersections of double cosets in algebraic groups, Indag. Math. (N.S.) 3 (1) (1992) 69–77.
[10] W. van der Kallen, Lectures on Frobenius splittings and B-modules, published for the Tata Institute of Fundamental Research, Bombay, 1993, Notes by S.P. Inamdar.